2 Skupovi i relacije

Pojam skupa je osnovni pojam u matematici pa se zato i ne definiše. Ovaj pojam objašnjavamo navodeći primjer nekog skupa i ukazujući na pravila njegove upotrebe u matematici.
Skupove označavamo velikim slovima latinice $A, B, C, ..., X, Y, ...$ a elemente nekog skupa malim slovima latinice $a, b, c, ..., x, y, ...$
Za skupove A i B kažemo da su jednaki ako su sastavljeni od istih elemenata.
Za skup A kažemo da je podskup skupa b ako svaki element iz skupa A istovremeno pripada i skupu B. To označavamo sa $A \subseteq B$.
Operacije sa skupovima definišemo sa:

$$\begin{aligned}
A \cap B &= \{x : x \in A \wedge x \in B\} \\
A \cup B &= \{x : x \in A \vee x \in B\} \\
A \backslash B &= \{x : x \in A \wedge x \notin B\} \\
A \Delta B &= (A \backslash B) \cup (B \backslash A) \\
A \times B &= \{(a, b) : a \in A \wedge b \in B\}
\end{aligned}$$

Zadatak 2.1 *Dokazati:*

$$(A \cup B)^C = A^C \cap B^C.$$

Rješenje:

$$\begin{aligned}
x &\in (A \cup B)^C \\
&\Longleftrightarrow x \notin A \cup B \\
&\Longleftrightarrow \rceil (x \in A \cup B) \\
&\Longleftrightarrow \rceil (x \in A \vee x \in B) \\
&\Longleftrightarrow x \notin A \wedge x \notin B \\
&\Longleftrightarrow x \in A^C \wedge x \in B^C \\
&\Longleftrightarrow x \in A^C \cap B^C.
\end{aligned}$$

Zadatak 2.2 *Dokazati:*

$$A \backslash B = A \cap B^C.$$

Rješenje:

$$\begin{aligned}
x &\in A \backslash B \\
&\Longleftrightarrow x \in A \wedge x \notin B \\
&\Longleftrightarrow x \in A \wedge x \in B^C \\
&\Longleftrightarrow x \in A \cap B^C.
\end{aligned}$$

Zadatak 2.3 *Dokazati*

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Rješenje:

$$\begin{aligned}
x &\in (A \cap B) \cup C \\
&\Longleftrightarrow x \in A \cap B \vee x \in C \\
&\Longleftrightarrow x \in A \wedge x \in B \vee x \in C \\
&\Longleftrightarrow x \in A \vee x \in C \wedge x \in B \vee x \in C \\
&\Longleftrightarrow (x \in A \vee x \in C) \wedge (x \in B \vee x \in C) \\
&\Longleftrightarrow x \in A \cup C \wedge x \in B \cup C \\
&\Longleftrightarrow x \in (A \cup C) \cap (B \cup C).
\end{aligned}$$

Zadatak 2.4 *Dokazati:*

$$(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C).$$

Rješenje:

$$\begin{aligned}
x &\in (A \cap B) \backslash C \\
&\Longleftrightarrow x \in A \cap B \wedge x \notin C \\
&\Longleftrightarrow x \in A \wedge x \in B \wedge x \notin C \\
&\Longleftrightarrow x \in A \wedge x \notin C \wedge x \in B \wedge x \notin C \\
&\Longleftrightarrow (x \in A \wedge x \notin C) \wedge (x \in B \wedge x \notin C) \\
&\Longleftrightarrow x \in A \backslash C \wedge x \in B \backslash C \\
&\Longleftrightarrow x \in (A \backslash C) \cap (B \backslash C).
\end{aligned}$$

Zadatak 2.5 *Dokazati:*

$$(A \backslash B) \cap (C \backslash D) = (A \cap C) \backslash (B \cup C).$$

Rješenje:

$$\begin{aligned}
x &\in (A \backslash B) \cap (C \backslash D) \\
&\Longleftrightarrow x \in A \backslash B \wedge x \in C \backslash D \\
&\Longleftrightarrow x \in A \wedge x \notin B \wedge x \in C \wedge x \notin D \\
&\Longleftrightarrow x \in A \wedge x \in C \wedge x \notin B \wedge x \notin D \\
&\Longleftrightarrow (x \in A \wedge x \in C) \wedge (x \notin B \wedge x \notin D) \\
&\Longleftrightarrow x \in A \cap C \wedge x \notin B \cup D \\
&\Longleftrightarrow x \in (A \cap C) \backslash (B \cup C).
\end{aligned}$$

Zadatak 2.6 *Dokazati:*

$$(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D).$$

Rješenje:

$$\begin{aligned}(x,y) &\in (A \cap B) \times (C \cap D)\\ &\iff x \in A \cap B \wedge y \in C \cap D\\ &\iff x \in A \wedge x \in B \wedge y \in C \wedge y \in D\\ &\iff x \in A \wedge y \in C \wedge x \in B \wedge y \in D\\ &\iff (x \in A \wedge y \in C) \wedge (x \in B \wedge y \in D)\\ &\iff (x,y) \in A \times C \wedge (x,y) \in B \times D\\ &\iff (x,y) \in (A \times C) \cap (B \times D).\end{aligned}$$

Zadatak 2.7 *Dat je skup* $X = \{1,2,3,4\}$. *Napisati relacije definisane sa*

$$\begin{aligned}\rho_1 &= \left\{(x,y) \in X^2 : x < y\right\}\\ \rho_2 &= \left\{(x,y) \in X^2 : x \leq y\right\}\\ \rho_3 &= \left\{(x,y) \in X^2 : x > y\right\}.\end{aligned}$$

Rješenje:
Kako je

$$\begin{aligned}X^2 &= X \times X = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),\\ &\quad (3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}\end{aligned}$$

tada je

$$\begin{aligned}\rho_1 &= \left\{(x,y) \in X^2 : x < y\right\} =\\ &= \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}.\\ \rho_2 &= \left\{(x,y) \in X^2 : x \leq y\right\} =\\ &= \{(1,1),(1,2),(1,3),(1,4),(2,2),(2,3),(2,4),(3,3),(3,4),(4,4)\}.\\ \rho_3 &= \left\{(x,y) \in X^2 : x > y\right\} =\\ &= \{(2,1),(3,1),(3,2),(4,1),(4,2),(4,3)\}.\end{aligned}$$

Zadatak 2.8 *Neka je*$X = \{1,2,3\}$. *Napisati elemente relacije definisane sa*

$$\rho = \left\{(x,y,z) \in X^3 : x + y \leq z\right\}.$$

Rješenje:
Kako je

$$\begin{aligned}X^3 &= X \times X \times X = \{(1,1,1),(1,1,2),(1,1,3),(1,2,1),(1,2,2),\\ &\quad (1,2,3),(1,3,1),(1,3,2),(1,3,3)\,(2,1,1),(2,1,2),\\ &\quad (2,1,3),(2,2,1),(2,2,2),(2,2,3),(2,3,1),(2,3,2),\\ &\quad (2,3,3)\,(3,1,1),(3,1,2),(3,1,3),(3,2,1),(3,2,2),\\ &\quad (3,2,3),(3,3,1),(3,3,2),(3,3,3)\}\end{aligned}$$

pa je

$$\begin{aligned}\rho &= \left\{(x,y,z) \in X^3 : x + y \leq z\right\}\\ &= \{(1,1,2),(1,1,3),(1,2,3)\}.\end{aligned}$$

Zadatak 2.9 *U skupu* $\mathbb{N}$ *definisana je relacija*

$$\rho = \left\{(x,y) \in \mathbb{N}^2 : x + 5y = 25\right\}.$$

Napisati elemente relacije ρ.

Rješenje:
Iz

$$\begin{aligned}x + 5y &= 25\\ x &= 25 - 5y\\ x &= 5\,(5 - y)\end{aligned}$$

vidimo da $y \in \{1,2,3,4\}$, jer u protivnom x ne bi bio prirodan broj. Sada

$$\begin{aligned}y &= 1 \Longrightarrow x = 20\\ y &= 2 \Longrightarrow x = 15\\ y &= 3 \Longrightarrow x = 10\\ y &= 4 \Longrightarrow x = 5\end{aligned}$$

pa je

$$\begin{aligned}\rho &= \left\{(x,y) \in \mathbb{N}^2 : x + 5y = 25\right\} =\\ &= \{(20,1),(15,2),(10,5),(5,4)\}.\end{aligned}$$